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OPTIMAL SETS FOR GAMES OVER THE SQUARE

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Summary: It is shown that every
non-void weak* closed convex set
of distributions on the unit
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timal strategies for one player
in some game with continuous pay-
off over the square.

OPTIMAL SETS FOR GAMES OVER THE SQUARE

I. Glicksberg and O. Gross

In this note we shall call the set of all optimal
strategies for one player in a game an optimal set. Optimal
sets, in two-person zero-sum games with continuous payoff
have the obvious properties of convexity and closure in the
weak* topology, so that one might ask whether these pro-
perties characterize such sets. The answer, in the case of
compact <u>metric</u> pure strategy spaces, is in the affirmative,
as will be shown.

We shall begin with games over the square. Since the
space $C(0,1)$, of continuous functions over the unit interval,
is separable, the space S of distributions over the unit
interval is, in the weak* (w^*) topology, compact and metric.

Let us set $(\varphi, f) = \int_0^1 \varphi(x) df(x)$ for $\varphi \in C(0,1)$ and f a

distribution, so that $\left\{ f \mid (\varphi, f) \geq \varepsilon \right\}$ is a w^* closed half

space of distributions.[1] The fundamental fact that we need

is that any non-void w^* closed convex set K of distributions

is the intersection of a sequence of such half spaces. This

can be established via Brouwer's Reduction Theorem (cf.

Hurewicz and Wallman, Dimension Theory, Index) which states

the following: In a space satisfying the second axiom of

countability a family of closed sets, with the property that

every decreasing sequence of elements of the family has its

intersection in the family, contains a minimal (or irreducible)

element. S certainly satisfies the second axiom of countability

as a compact metric space, and if we take as our family of

closed sets all countable intersections of half spaces which

contain K, we clearly may apply the theorem to obtain a

minimal set K_o which is the intersection of a sequence of

half spaces and contains K. But certainly it can be no

larger than K, for if f is in K_o but not in K then we have

a closed half space H containing K but not f, so that

$K_o \cap H$ is in the family and properly contained in K_o contra-

dicting the minimal character of K_o. Thus we may write

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[1] We may clearly take such functions φ and constants a
to be bounded by 1, as we shall do below.

$$K = \bigcap_n \left\{ f \,|\, f \in S \text{ and } (\varphi_n, f) \geqq a_n \right\}.$$

We may now construct a game over the square having K as the optimal set for, say, the maximizing player. We set

$$M(x, \tfrac{1}{n}) = \tfrac{1}{n}(\varphi_n(x) - a_n), \qquad n = 1, 2, \ldots,$$

and extend M to a continuous function over the square by setting $M(x,0) = 0$ and interpolating linearly along vertical lines. Since K is non-void, we have clearly

$$\int M(x, \tfrac{1}{n}) df(x) = \tfrac{1}{n} \left[(\varphi_n, f) - a_n \right] \geqq 0$$

for all n, and thus for all y

$$\int M(x, y) df(x) \geqq 0$$

for $f \in K$. Since $M(x,0) = 0$, 0 is the value of the game, and we see that a distribution f is optimal only if $(\varphi_n, f) - a_n \geqq 0$ for all n, or that f is in K. Thus K is the optimal set for the maximizing player in this game.

In the case of a metric pure strategy space A for the maximizing player the procedure is similar, inasmuch as $C(A)$ is again separable (this is only true when A is metric, so that the method used for the square can only extend to this case) and again we obtain our set K as the intersection

of a sequence of half spaces. In order to construct the appropriate payoff we must require the minimizing player's pure strategy space B (which we assume compact though not necessarily metric) be infinite. (Indeed if B contains just n points then the maximizing player always may play just n points and some sets K will not be optimal sets.)

Since B is infinite, at most one point y_0 in B has the property that each of its open neighborhoods has a finite complement (if we had two such points we would have disjoint neighborhoods with finite complements, and the space B would be finite). Choose a point $y_1 \neq y_0$ and an open neighborhood W_1 of y_1 for which W_1' (the complement of W_1) is infinite, further, a neighborhood V_1 of y_1 for which $\overline{V_1} \subset W_1$ (B is of course regular). Since W_1' is an infinite compact space we may find by the same argument a point $y_2 \in W_1'$ with a relative open neighborhood $W_2 \cap W_1'$ (where W_2 is open in B) with infinite complement in W_1', and thus by the regularity of B an open neighborhood V_2 of y_2 for which $\overline{V_2} \cap \overline{V_1} = 0$. The relative complement in W_1' of $W_2 \cap W_1'$ is of course $W_1' \cap W_2' = (W_1 \cup W_2)'$, again an infinite compact space, and thus by induction we obtain a sequence (y_n) of points and (V_n) of pairwise disjoint open neighborhoods.

Let $y_n \in U_n$, $\bar{U}_n \subset V_n$ where U_n is open, and let h_n be a
continuous function, $0 \leq h_n \leq 1$, $h_n(y_n) = 1$, which vanishes
outside U_n, and define M by

$$M(x,y) = \sum \frac{1}{n} h_n(y) \left[\varphi_n(x) - a_n \right],$$

where $K = \bigcap \left\{ f \mid (\varphi_n, f) \geq a_n \right\}$. Since at most one term of
the sum differs from zero M is well defined; it is easily
seen to be continuous. For given $\varepsilon > 0$ and a point
$(x_0, y_0) \in A \times B$, if $y_0 \in \bigcup V_n$ then $y_0 \in V_k$, and, in $A \times V_k$, the
function is $\frac{1}{k} h_k(y) \left[\varphi_k(x) - a_k \right]$ and neighborhoods
U of x_0 and V of y_0 exist for which

$$\lvert M(x,y) - M(x_0, y_0) \rvert < \varepsilon \quad \text{for } x \in U, \; y \in V,$$

by virtue of the continuity of h_k and φ_k. If $y_0 \notin \bigcup V_k$
then y_0 has for each n a neighborhood $V^{(n)}$ for which
$V^{(n)} \cap \bar{U}_k = 0$, $k = 1, \ldots, n$, so that

$$\lvert M(x,y) \rvert < \frac{1}{n} \quad \text{for } x \in A, \; y \in V^{(n)}$$

whence M is continuous.

Since B is compact, $\{y_n\}$ has at least one cluster
point y_0, and clearly we must have $M(x, y_0) = 0$ for all x.
Since $\int M(x,y) df(x) = \sum \frac{1}{n} h_n(y) \left[(\varphi_n, f) - a_n \right] \geq 0$ for
$f \in K$, the value is 0, and f is optimal only if

$$\int M(x,y_n)df(x) = \frac{1}{n}\left[(\varphi_n,f) - a_n\right] \geqq 0$$

or $f \in K$.

In summary: If the pure strategy space of the max-imizing player is compact metric and that of his opponent is compact and infinite, every w^* closed convex set of his mixed strategies is an optimal set for some continuous payoff.

Unfortunately, this does not have an obvious bearing on the important problem of determining structure of solution sets, i.e. the pair of optimal sets in a game. Not all possible pairs of optimal sets, for example, are attainable. Consider the following special case on the square. Let the optimal set for the maximizing player be the uniform distribution, $f(x) = x$, and that of the min-imizing player be the set of all distributions. Suppose M is the continuous payoff of a game with this solution set, and let the value of the game be v. Since any pure strategy is optimal for the minimizing player $M(x,y) \leqq v$ for all x,y. If $M(x,y) < v$ for some point in the square, then obviously this holds in a neighborhood and $\int M(x,y)dx < v$, which contradicts the optimality of f. Therefore $M \equiv v$, and all distributions are optimal for the maximizing player, a contradiction.